Paper: Ordinary Differential Equations
Lesson: Existence and Uniqueness of Solutions of first order differential equations

Lesson Developer: Dr. Sada Nand Prasad
College/Department: Department of Mathematics, A.N.D. College, University of Delhi

Paper: Ordinary Differential Equations

Lesson: Existence and Uniqueness of Solutions of first order differential equations

Course Developer: Dr. Sada Nand Prasad

College/Department: Department of Mathematics, A.N.D. College (D.U.)

	Author(s)	**Reviewer**
Name	Dr. Sada Nand Prasad	Dr. Chaman Singh
College/ Department:	Acharya Narendra Dev College (D.U.)	Acharya Narendra Dev College (D.U.)
Contact No:	9711969937	9953579621
E-mail id:	prasadsadanand@yahoo.co.in	chamaniimt@gmail.com

Table of Contents:

1. Learning Outcomes:

After completing this chapter, we can

- resolve the differential equations into rational and solve it.
- solve equations for p, x and y.
- explain Clairaut's equation and find its solution
- find the singular solution of a differential equation
- Find the envelopes and orthogonal trajectories of the family of different surfaces
- state the existence and uniqueness theorem for first order ordinary differential equations and use it to know whether the solution exist and unique or not.

2. Introduction:

We have used many methods to solve differential equations of first order and of degree 1, e.g, differential equations that can be separated in different variables, exact differential equations, equations that can be reduced to homogeneous equation and those equations that become exact when we multiply them by some integrating factor.

In this chapter, we will keep our discussion on the differential equations which are of first order but of higher degree.

Let us take $\frac{dy}{dx} = p$, then the general form of the first order and nth degree differential equation is given by the equation

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + \dots + A_n = 0 \qquad (1)$$

where A_1, A_2,..., A_n are functions of x and y.

It is not simple to find the solution of eqn. (1) in its general form. In this chapter we consider only that type of eqn. (1) which we can solve easily and explain the methods for solving those equations.

Also we shall discuss the Clairaut equation, the singular solution of the differential equations, the envelopes and orthogonal trajectories of the family of surfaces and state and the existence and uniqueness theorem for first order ordinary differential equations in this chapter.

3. Equations which can be factorized:

The general form of the first order and nth degree differential equation is given by,

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + \dots + + A_n = 0$$

where A_1, A_2,..., A_n are functions of the variables x and y.

Now there are two possibilities:

(a) If we resolve $p^n + A_1 p^{n-1} + A_2 p^{n-2} + \dots + A_n$ into rational factors of degree 1, then it can be written as

$$(p - f_1)(p - f_2)\dots(p - f_n) = 0 \qquad (2)$$

where f_1, f_2,..., f_n are functions of the variables x and y.

Since all those values of y, for which the factors in eqn. (2) become zero, will satisfy eqn. (1). Hence, to solve eqn. (1), we will have to equate each of the factors given in eqn. (2) to zero. i.e.,

$$p - f_r = 0, r = 1, 2, ..., n \qquad (3)$$

If

$$F_r(x, y, C_r) = 0, r = 1, 2, ..., n \qquad (4)$$

where C_r, $r = 1, 2, ..., n$ *are arbitrary* constants.

are the solutions for eqns. (3), then the general solution of eqn. (1) is given by

$$F_1(x, y, C_1).F_2(x, y, C_2)...F_n(x, y, C_n) = 0, \qquad (5)$$

All the constants in eqn. (5), namely, $C_1, C_2, ..., C_n$ can have infinite number of values, so all these solutions given by eqn. (5), will remain general even if we take $C_1 = C_2 = ... = C_n = C$. Hence, the general solution is given by

$$F_1(x, y, C).F_2(x, y, C)...F_n(x, y, C) = 0, \qquad (6)$$

(b) When the left-hand side of eqn. (1) cannot be factorized. We will take this possibility in the next section.

Value Additions:
Since we are dealing with the first order differential equation, the general solution should contain only one arbitrary constant. There is no loss of generality by replacing the n arbitrary constants by a single arbitrary constant.

Example 1: Solve the differential equation $x^2 p^2 + x y p - 6 y^2 = 0, where\ p = \frac{dy}{dx}$.

Solutions: We have

$$x^2 p^2 + x y p - 6 y^2 = 0 \Rightarrow p = \frac{-x y \pm \sqrt{x^2 y^2 + 24 x^2 y^2}}{2} = \frac{2y}{x}\ or\ \frac{-3y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2y}{x}\ or\ \frac{dy}{dx} = \frac{-3y}{x} \Rightarrow \frac{dy}{y} = \frac{2dx}{x}\ or\ \frac{dy}{y} = \frac{-3dx}{x}$$

Integrating, we get,

$$\log y = 2\log x + \log C_1 \ or \log y = -3\log x + \log C_2$$

$$i.e., y = C_1 x^2 \ or \ y = \frac{C_2}{x^3}$$

Hence the general solution is given by

$$\left(y - C x^2\right)\left(y - \frac{C}{x^3}\right) = 0$$

$$i.e., \left(y - C x^2\right)\left(y x^3 - C\right) = 0$$

Example 2: Solve the differential equation

$$\left(x + 2y\right)p^3 + 3\left(x + y\right)p^2 + \left(2x + y\right)p = 0, where \ p = \frac{dy}{dx}.$$

Solution: We have

$$\left(x + 2y\right)p^3 + 3\left(x + y\right)p^2 + \left(2x + y\right)p = 0 \Rightarrow p\left[\left(x + 2y\right)p^2 + 3\left(x + y\right)p + \left(2x + y\right)\right] = 0$$

$$\Rightarrow \quad p\left[\left(x + 2y\right)p^2 + p\left\{\left(x + 2y\right) + \left(2x + y\right)\right\} + \left(2x + y\right)\right] = 0$$

$$\Rightarrow \quad p\left[\left(x + 2y\right)p\left(p + 1\right) + \left(2x + y\right)\left(p + 1\right)\right] = 0 \Rightarrow p\left[\left(p + 1\right)\left\{\left(x + 2y\right)p + \left(2x + y\right)\right\}\right] = 0$$

$$\Rightarrow \quad p\left(p + 1\right)\left[\left(x + 2y\right)p + \left(2x + y\right)\right] = 0$$

$$\Rightarrow \quad p = 0, p + 1 = 0 \ and \left(x + 2y\right)p + \left(2x + y\right) = 0$$

Solving these equations, we get,

$$y = C_1, \frac{dy}{dx} + 1 = 0 \Rightarrow x + y = C_2 \ and \left(x + 2y\right)dy + \left(2x + y\right)dx = 0$$

$$x\,dy + y\,dx + d\left(x^2 + y^2\right) = 0 \ which \ on \ \text{integration gives}$$

$$\text{x y} + x^2 + y^2 = C_3$$

Hence, the general solution of the given equation is given by

$$\left(y - C\right)\left(x + y - C\right)\left(x y + x^2 + y^2 - C\right) = 0.$$

Example 3: Solve the differential equation $\left(p - 2x\right)\left(p - y\right) = 0$.

Solution: We have

$$\left(p - 2x\right)\left(p - y\right) = 0 \Rightarrow \frac{dy}{dx} = 2x \ and \ \frac{dy}{dx} = y \ or \ \frac{dy}{y} = dx$$

$$Integrating, we \ get, \ y = x^2 + C_1 \ and \ \log y + \log C_2 = x \Rightarrow y C_2 = e^x$$

Hence the general solution is given by $\left(y-x^2-C\right)\left(Cy-e^x\right)=0$.

Example 4: Find the general solution of the equation $p^2+px+py+xy=0$.

Solution: We have

$$p^2+px+py+xy=0 \Rightarrow (p+x)(p+y)=0 \Rightarrow \frac{dy}{dx}=-x \, and \, \frac{dy}{dx}=-y \, or -\frac{dy}{y}=dx$$

$$Integrating, we\, get, y=-\frac{x^2}{2}+C_1 \, and -\ln|y|+C_2=x$$

Hence the general solution is given by $\left(2y+x^2-C\right)\left(x+\ln|y|-C\right)=0$.

I.Q. 1

I.Q. 2

I.Q. 3

I.Q. 4

4. Equations which cannot be factorized:

Let us write eqn. (1) in the form of

$$f(x,y,p)=0 \qquad (7)$$

Then, we cannot solve eqn. (7) in its general form. It may be solvable for x, y, p, or it may be of first degree in the variables x and y and can be solved by the methods discussed in the earlier chapters.

Let us now discuss these cases one by one.

4.1. Equations Solvable for x:

Let the equation is given by $x=f(y,p)$ (8)

Differentiate eqn. (8), with respect to y, we get,

$$\frac{1}{p}=\phi\left(y,p,\frac{dp}{dy}\right) \qquad (9)$$

When we solve this equation, we will get a relation between p and y which can be written in the form

$$f(y,p,c)=0 \qquad (10)$$

where c is an arbitrary constant.

Eliminating p from the eqns. (8) and (10), we get the required solution. If we cannot eliminate p from eqns. (8) and (10), then we obtain x and y in p and all these together provide the required solution.

Example 5: Solve the differential equation $y=2px-p^3y^2$

Solution: we have $y=2px-p^3y^2 \Rightarrow x=\dfrac{y}{2p}+\dfrac{p^2y^2}{2}$

Differentiate the above eqn. with respect to y, we get,

$$\frac{1}{p}=\frac{1}{2}\left(\frac{1}{p}-\frac{y}{p^2}\frac{dp}{dy}\right)+\frac{2p\frac{dp}{dy}y^2}{2}+\frac{2p^2}{2}\cdot y$$

$$\Rightarrow \frac{1-2p^3y}{p}=\frac{dp}{dy}y\frac{\left(2p^3y-1\right)}{p^2}\Rightarrow \frac{dp}{dy}y=-p$$

$$i.e., \frac{dp}{p}+\frac{dy}{y}=0, Integrating\ we\ get, \log p+\log y=\log C, i.e., \mathrm{p\,y=C}$$

Eliminating p from this equation and $y=2px-p^3y^2$, we get,

$$y=\frac{2C}{y}x-\frac{C^3}{y^3}y^2 \Rightarrow y^2=2Cx-C^3$$

Example 6: Solve the differential equation $p^2-py+x=0$

Solution: we have $p^2-py+x=0 \Rightarrow x=py-p^2$

Differentiate the above eqn. with respect to y, we get,

$$\frac{1}{p}=p+y\frac{dp}{dy}-2p\frac{dp}{dy}\Rightarrow \frac{1-p^2}{p}=\frac{dp}{dy}\left(y-2p\right)\Rightarrow \frac{dy}{dp}-\frac{p}{1-p^2}y=-\frac{2p^2}{1-p^2}$$

It is a linear equation with y as dependent variable, so we will solve this equation after getting the integrating factor.

$$I.F.=e^{-\int\frac{p}{1-p^2}dp}=e^{\frac{1}{2}\ln\left(1-p^2\right)}=\sqrt{1-p^2}$$

$$\Rightarrow y\sqrt{1-p^2}=-\int\frac{2p^2}{\sqrt{1-p^2}}dp+C$$

$$putting\ p=\sin\theta\ gives \int\frac{2p^2}{\sqrt{1-p^2}}dp=\sin^{-1}p-p\sqrt{1-p^2}$$

$$\therefore y=\frac{\sin^{-1}p}{\sqrt{1-p^2}}+p+\frac{C}{\sqrt{1-p^2}}$$

Thus x and y given by $x=py-p^2$ and $y=\frac{\sin^{-1}p}{\sqrt{1-p^2}}+p+\frac{C}{\sqrt{1-p^2}}$ will constitute the solution.

Example 7: Find the general solution of the equation $p^2y+2px-y=0.$.

Solution: We have

$$p^2y+2px=y \Rightarrow x=\frac{y}{2p}-\frac{py}{2} \Rightarrow \frac{dx}{dy}=\frac{1}{p}=\frac{1}{2p}+\frac{y}{2}\left(-\frac{1}{p^2}\right)\frac{dp}{dy}-\frac{y}{2p}-\frac{p}{2}-\frac{y}{2}\frac{dp}{dy}$$

$$\Rightarrow \frac{1+p^2}{2p}\left(1+\frac{y}{p}\frac{dp}{dy}\right)=0 \Rightarrow Either\ \frac{1+p^2}{2p}=0 \Rightarrow p^2=-1, solution\ is\ imaginary$$

$$or, 1+\frac{y}{p}\frac{dp}{dy}=0 \Rightarrow \frac{1}{y}+\frac{1}{p}\frac{dp}{dy}=0$$

$$Integrating, we\ get,\ \log y+\log p=\log c \Rightarrow y\,p=c$$

$$putting\ value\ of\ p\ in\ the\ eqn.\ x=\frac{y}{2p}-\frac{py}{2}, we\ get, x=\frac{y^2}{2c}-\frac{c}{2}$$

Value Addition:
When eqn. (7) is solvable for x, we need to differentiate it with respect to y, whereas, when eqn. (7) is solvable for y, we need differentiate it with respect to x.

I.Q. 5

I.Q. 6

I.Q. 7

4.2. Equations Solvable for y:

Let the equation is given by $y=f(x,p)$ (11)

Differentiate eqn. (11), with respect to x, we get,

$$p=\phi\left(x, p, \frac{dp}{dx}\right) \qquad (12)$$

When we solve this equation, we will get a relation between p and x which can be written in the form

$$f(x,p,c)=0 \qquad (13)$$

where c is an arbitrary constant.

Now, eliminating p between eqns. (11) and (13), we get the required solution. If we cannot eliminate p between eqns. (11) and (13), then we obtain x and y in p and all these together provide the required solution, as we have done in the last section.

Example 8: Solve the differential equation $p^3+p-e^y=0$

Solution: we have

$$p^3+p-e^y=0 \Rightarrow p^3+p=e^y$$
$$Taking\ \log both\, sides, we\, get,$$
$$y=\ln p+\ln\left(p^2+1\right)$$

Differentiate the above eqn. with respect to x, we get,

$$p=\frac{1}{p}\frac{dp}{dx}+\frac{2p}{p^2+1}\frac{dp}{dx}\Rightarrow dx=\left(\frac{1}{p^2}+\frac{2}{p^2+1}\right)dp$$
$$Integrating, we\, get, x=-\frac{1}{p}+2\tan^{-1}p+C$$

Thus x and y given by $x=-\frac{1}{p}+2\tan^{-1}p+C$ and $y=\ln p+\ln\left(p^2+1\right)$ will constitute the solution.

Example 9: Find the solution of the differential equation $y=2px+p^2y$ for y.

Solution: we have

$$y=2px+p^2y \Rightarrow y\left(1-p^2\right)=2px\, i.e, y=\frac{2p}{1-p^2}x$$

Differentiate the above eqn. with respect to x, we get,

$$p=\frac{2p}{1-p^2}+2x\frac{1+p^2}{\left(1-p^2\right)^2}\frac{dp}{dx}\Rightarrow p\left(1-\frac{2}{1-p^2}\right)=2x\frac{1+p^2}{\left(1-p^2\right)^2}\frac{dp}{dx}$$

$$i.e., -p\frac{\left(1+p^2\right)}{1-p^2}=2x\frac{1+p^2}{\left(1-p^2\right)^2}\frac{dp}{dx}\Rightarrow -p=\frac{2x}{1-p^2}\frac{dp}{dx}\Rightarrow\frac{2}{p\left(1-p^2\right)}dp+\frac{dx}{x}=0$$

$$Integrating, we\, get, \int\left(\frac{2}{p}+\frac{1}{1-p}-\frac{1}{1+p}\right)dp+\log x=\log C$$

or $\quad 2\log p-\log(1-p)-\log(1+p)+\log x=\log C \Rightarrow \frac{p^2 x}{1-p^2}=C$

Eliminating p from the above equation and $y=2px+p^2 y$, we get

$$y=\frac{2\sqrt{C}\,x}{\sqrt{x+C}}+\frac{C y}{x+C} \Rightarrow y-\frac{C y}{x+C}=\frac{2\sqrt{C}\,x}{\sqrt{x+C}} \Rightarrow \frac{xy+Cy-Cy}{x+C}=\frac{2\sqrt{C}\,x}{\sqrt{x+C}}$$

$$i.e., y=\left(2\sqrt{C}\right)\left(\sqrt{x+C}\right) \Rightarrow y^2=4Cx+4C^2$$

Example 10: Find the general solution of the equation $y=2px+p^4x^2$..

Solution: We have

$$y=2px+p^4x^2 \text{ where } p=\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=p=2p+2x\frac{dp}{dx}+2xp^4+4x^2p^3\frac{dp}{dx}$$

$$\Rightarrow \left(1+2xp^3\right)\left(p+2x\frac{dp}{dx}\right)=0 \Rightarrow Either\, 1+2xp^3=0$$

$$or,\ p+2x\frac{dp}{dx}=0 \Rightarrow \frac{1}{x}+\frac{2}{p}\frac{dp}{dx}=0$$

$$Integrating, we\ get,\ \log x+2\log p=\log c \Rightarrow xp^2=c$$

$$putting\ value\ of\ p\ in\ the\ eqn.\ y=2px+p^4x^2, we\ get, y=2\sqrt{cx}+c^2$$

Hence the general solution is given by $y=2\sqrt{cx}+c^2$.

I.Q. 8

I.Q. 9

I.Q. 10

5. Clairaut's Equation:

The equation $y = px + f(p)$ is known as Clairaut's form. Differentiating the equation $y = px + f(p)$ with respect to x, we get,

$$\frac{dy}{dx}=p+x\frac{dp}{dx}+f'(p)\frac{dp}{dx} \Rightarrow p=p+x\frac{dp}{dx}+f'(p)\frac{dp}{dx}$$

$$i.e., \frac{dp}{dx}\left(x+f'(p)\right)=0, \therefore Either\ x+f'(p)=0\ or\ \frac{dp}{dx}=0$$

$$When\ \frac{dp}{dx}=0, p=C, and\ hence\ y=Cx+f(C)$$

$x+f'(p)=0$ will give a solution which is known as singular solution, we will discuss in the next section.

Value Addition:
Note: Eqn. $y=Cx+f(C)$ is an equation of a family of straight lines. Thus, the general solution of the Clairaut's equation is an equation of a family of straight lines.

Example 11: Solve the equation $(px-y)(px+y)=2p$.

Solution: Substituting $x^2=X, y^2=Y \, and \, P=\frac{dY}{dX}=\frac{2y\frac{dy}{dx}}{2x} \Rightarrow P=\frac{yp}{x}$ in the equation $(px-y)(px+y)=2p$, we get,

$$\left(\frac{x^2P}{y}-y\right)(Px+x)=\frac{2xP}{y} \Rightarrow (PX-Y)(P+1)=2p, \therefore Y=PX-\frac{2P}{P+1}$$

Which is a Clairaut's equation.

Hence the solution is $Y=CX-\frac{2C}{C+1} \Rightarrow y^2=Cx^2-\frac{2C}{C+1}$.

Example 12: Solve the equation $y=xp+\frac{1}{p}$.

Solution: We have, $y=xp+\frac{1}{p}$ which is already a Clairaut's equation with

$$f(p)=\frac{1}{p} \, and \, f'(p)=-\frac{1}{p^2}$$

Hence the solution is $y=Cx+\frac{1}{C}, C\neq 0, is\,an\,arbitrary\,contt.$ Also,

$$x+f'(p)=0 \Rightarrow x-\frac{1}{p^2}=0$$

Eliminating p between the equation $x-\frac{1}{p^2}=0$ and the equation $y=xp+\frac{1}{p}$ yields $y=x\frac{1}{\sqrt{x}}+\sqrt{x}=2\sqrt{x}\Rightarrow y^2=4x$

which is the required singular solution of the equation $y=xp+\frac{1}{p}$.

Example 13: Find the singular solution of the equation $p^2+4xp-4y=0$.

Solution: We have

$$p^2+4xp-4y=0\Rightarrow y=px+\frac{p^2}{4}\Rightarrow p=p'x+p+\frac{p}{2}p'\Rightarrow p'\left(x+\frac{p}{2}\right)$$

$$Either\ p'=0\Rightarrow p=c, or\ x+\frac{p}{2}=0\Rightarrow p=-2x\, or, c=-2x$$

$$putting\ p=c\, in\ y=px+\frac{p^2}{4}, we\ get\ y=cx+\frac{c^2}{4}\Rightarrow y=(-2x)x+\frac{(-2x)^2}{4}=-x^2$$

Now the arbitrary constant is absent. Also y satisfies eqn. $y=px+\frac{p^2}{4}$, therefore, the singular solution of eqn. $y=px+\frac{p^2}{4}$.

Example 14: Find the singular solution of the Clairaut's equation $y=px+p^2$.

Solution: We have the clairaut's equation $y=px+p^2$

Differentiate the above equation with respect to x, we get,

$$p=p+xp'+2pp'\Rightarrow p'(x+2p)=0$$

$$\Rightarrow Either\ p'=0\Rightarrow p=c$$

$$or\ x+2p=0\Rightarrow p=-\frac{x}{2} or\ c=-\frac{x}{2}$$

$$putting\ p=c\, in\, y=px+p^2\ gives\ y=cx+c^2\Rightarrow y=-\frac{x}{2}x+\frac{x^2}{4}\Rightarrow y=-\frac{x^2}{4}$$

Now the arbitrary constant is absent. Also y satisfies eqn. $y=cx+c^2$, therefore, the singular solution of eqn. $y=cx+c^2$.

I.Q. 11

I.Q. 12

I.Q. 13

6. Singular Solution:

Suppose the general solution of an ordinary differential equation

$$f(x, y, p) = 0 \text{ is } F(x, y, C) = 0 \qquad (14).$$

A singular solution is defined as neither the general solution nor a particular solution of a differential equation.

It is proved that a curve representing the singular solution is the envelope of the curves representing the general solution of the differential equation.

So to get the singular solution, find the envelope of the curves representing the general solution of the differential equation.

In the case of Clairaut's form $x + f'(p) = 0$ taken along with the equation $y = xp + f(p)$ on the elimination of p gives the singular solution.

Definition: C – discriminant:

Suppose the general solution of $f(x, y, p) = 0$ is $F(x, y, C) = 0$. The equation obtained by eliminating C from $F(x, y, C) = 0$ and $\frac{\partial F}{\partial C} = 0$ is called the C – discriminant of the equation.

Remark: If the general solution is a quadratic equation in C, i. e.,

A c^2+B c + D = 0, then the C- discriminant is B^2 = 4 A D.

p – discriminant:

The eliminant of p from $f(x, y, p) = 0$ and $\frac{\partial f}{\partial p} = 0$ is called the p – discriminant of the equation.

Remark: If the given differential equation can be expressed in the form

$L\,p^2 + M\,p + N = 0$, then the p- discriminant is $M^2 = 4\,L\,N$.

Value Addition:
Observation: (a) The envelope of F(x, y, C) = 0, is the part of C – discriminant. (b) The envelope of F(x, y, C) = 0, is the part of p – discriminant. (c) The envelope of F(x, y, C) = 0 satisfies the equation f(x, y, p) = 0. Thus a singular solution is obtained by the following method. (i) Find the C – discriminant of the given equation by eliminating C between F(x, y, C) = 0 and $\frac{\partial F}{\partial C}=0$. (ii) Find the p – discriminant of the given equation by eliminating p between f(x, y, p) = 0 and $\frac{\partial f}{\partial p}=0$. (iii) Take common factor between the C – discriminant and p – discriminant. (iv) Test whether the factors in (iii) satisfy the given equation. (v) Only those that satisfy f(x, y, p) = 0 will constitutes the singular solution.

Example 15: Solve the differential equation $y=-x\,p+x^4\,p^2$. Also find the singular solution.

Solutions: We have $y=-x\,p+x^4\,p^2$

Differentiate with respect to x, we get,

$$p=-p-x\frac{dp}{dx}+4x^3p^2+2x^4\,p\frac{dp}{dx}\Rightarrow 2p-4x^3p^2+x\frac{dp}{dx}-2x^4\,p\frac{dp}{dx}=0$$

$$i.e., 2p\left(1-2px^3\right)+x\frac{dp}{dx}\left(1-2px^3\right)=0\Rightarrow\left(2p+x\frac{dp}{dx}\right)\left(1-2px^3\right)=0$$

$$\therefore 2p+x\frac{dp}{dx}=0\Rightarrow\frac{2dx}{x}+\frac{dp}{p}=0, Integrating, we\ get, 2\log x+\log p=\log C\Rightarrow x^2\ p=C$$

Eliminating p from this equation and $y=-xp+x^4p^2$, we get,

$$y=-x\frac{C}{x^2}+C^2\Rightarrow C^2x-C-xy=0,$$

$$Differentiating\ with\ respect\ to\ C, we\ get, 2\mathrm{C}\,\mathrm{x}-1=0.$$

$$Putting\ C=\frac{1}{2x}\ in\ C^2x-C-xy=0, we\ get, \frac{x}{4x^2}-\frac{1}{2x}-xy=0\Rightarrow 1+4x^2\ y=0$$

Which is a C – discriminant. We can obtain the p – discriminant by eliminating p from $y=-xp+x^4p^2$ and $\frac{\partial f}{\partial p}=0=-x+2x^4\,p$, $i.e., p=\frac{1}{2x^3}$

Hence the p – discriminant is $y=-\frac{x}{2x^3}+\frac{x^4}{4x^6}\Rightarrow 4x^2\,y+1=0$. Since the common expression between p and C discriminant is $1+4x^2\,y=0$, therefore $1+4x^2\,y=0$ is a singular solution if it satisfy the equation $y=-\frac{1}{4x^2}\ and\ p=\frac{1}{2x^3}$

Substituting the value of y and p satisfies the equation $y=-xp+x^4p^2$, hence $1+4x^2\,y=0$ is a singular solution.

Example 16: If the general solution of the differential equation $x^2p^2+yp\left(2x+y\right)+y^2=0\ is\ C^2-4Cxy-2Cy^2+4x^2y^2=0,$ Find the singular solution.

Solution: The equation $x^2p^2+yp\left(2x+y\right)+y^2=0$ is quadratic in p, its p – discriminant is

$$y^2\left(2x+y\right)^2-4x^2y^2=0\Rightarrow y^3\left(4x+y\right)=0$$

The general solution

$C^2 - 4Cxy - 2Cy^2 + 4x^2y^2 = 0,$ is quadratic in C. Hence the C – discriminant is given by

$$\left(2y^2 + 4xy\right)^2 - 16x^2y^2 = 0, i.e., y^3\left(y + 4x\right) = 0.$$

The common factors are $y = 0$ *and* $y + 4x = 0.$ Since these factors satisfies the differential equation $x^2p^2 + yp\left(2x + y\right) + y^2 = 0$ hence these are the singular solutions.

Example 17: Solve the differential equation $y = px - p^2$. Also find the singular solution.

Solution: We have, $y = xp - p^2$ which is a Clairaut's equation with

$$f\left(p\right) = -p^2 \text{ and } f'\left(p\right) = -2p$$

Hence the solution is $y = Cx - C^2$, *is an arbitrary contt.*

Also, $x + f'\left(p\right) = 0 \Rightarrow x - 2p = 0$

The elimination of p from the above equation and the equation $y = xp - p^2$ yields $y = x\dfrac{x}{2} - \dfrac{x^2}{4} = \dfrac{x^2}{4} \Rightarrow x^2 = 4y$

which is a singular solution of the given equation.

Example 18: Solve the differential equation $x^2\left(y - px\right) = yp^2$

Solution: Substituting $x^2 = X, y^2 = Y \text{ and } P = \dfrac{dY}{dX} = \dfrac{2y\dfrac{dy}{dx}}{2x} \Rightarrow P = \dfrac{yp}{x}$ in the equation $x^2\left(y - \dfrac{Px^2}{y}\right) = y\dfrac{P^2x^2}{y^2}$, we get,

$$x^2\left(y - \frac{Px^2}{y}\right) = y\frac{P^2x^2}{y^2} \Rightarrow Y - PX = P^2 \Rightarrow Y = PX + P^2$$

Which is a Clairaut's equation.

Hence the solution is $Y = CX + C^2 \Rightarrow y^2 = Cx^2 + C^2$.

I.Q. 14

I.Q. 15

I.Q. 16

I.Q. 17

7. Envelops and Orthogonal trajectories:

Definition: **Orthogonal Trajectories:** Two families of curves, such that each member of one family cuts every member of the other family at right angles, are called **orthogonal trajectories** of one another.

From the equation $f(x, y, C) = 0$ representing one parameter family of curves, we can form a differential equation of the first order $F\left(x, y, \frac{dy}{dx}\right) = 0$ which is the differential equation of the family of curves. Replacing $\frac{dy}{dx}$ by

$-\frac{1}{\frac{dy}{dx}}$ in $F\left(x, y, \frac{dy}{dx}\right) = 0$, we get $F\left(x, y, -\frac{1}{\frac{dy}{dx}}\right) = 0 \qquad (15)$.

This is equation of the family of orthogonal trajectory.

For a family of curves by the polar equation $f(\mathrm{r}, \theta, c) = 0$ we can establish a differential equation of the family $F\left(\mathrm{r}, \theta, \frac{dr}{d\theta}\right) = 0$. The differential equation of the orthogonal trajectory is $F\left(\mathrm{r}, \theta, -\frac{r^2}{\frac{dr}{d\theta}}\right) = 0 \qquad (16)$

Thus if the equation of the family of curves be given, we shall first find the differential equation by differentiation and elimination of the parameter. Then find the differential equation of the trajectory by the above process and solve the differential equation and get the Cartesian or polar equation of the trajectory.

Example 19: Find the orthogonal trajectories of a family of circles which touches a given line at any given point.

Solution: If we take the given line as the x – axis and the given point where the line touches the circle as the origin, we get the equation of that family of circles as

$$x^2+y^2=2ky, \textit{differentiating w.r.to } x, \textit{we get}, 2x+2yp=2kp$$
$$\text{Eliminating } 2k \textit{ from these two equations, we get}, x^2+y^2=y\left(\frac{2x+2yp}{p}\right)$$
$$\Rightarrow\left(x^2-y^2\right)p-2xy=0$$

The orthogonal trajectory of the family of circles has the differential equation

$$\left(x^2-y^2\right)\left(\frac{-1}{p}\right)-2xy=0\Rightarrow\left(x^2-y^2\right)+2xyp=0$$
$$\textit{Putting } y=vx \textit{ in the above equation, we get}, \left(x^2-v^2x^2\right)p+2x^2v\left(v+x\frac{dv}{dx}\right)=0$$
$$\textit{i.e.}, x\frac{dv}{dx}=\frac{-1+v^2}{2v}-v\Rightarrow\frac{2v\,dv}{1+v^2}+\frac{dx}{x}=0, \textit{which on } \text{integration gives,}$$
$$\log\left(1+v^2\right)+\log x=\log C$$
$$\Rightarrow\left(1+v^2\right)x=C \textit{ i.e.}, \left(1+\frac{y^2}{x^2}\right)x=C\Rightarrow x^2+y^2=Cx \qquad (17)$$

The equation (17) represents a family of circles touching the y – axis and is the required equation of the orthogonal trajectory.

Example 20: Find the orthogonal trajectories of the family $r^n=a^n\sin n\theta$

Solution: Differentiating the equation $r^n=a^n\sin n\theta$ with respect to θ, we have

$$n\,r^{n-1}\frac{dr}{d\theta}=n\,a^n\cos n\theta \Rightarrow r^n\frac{dr}{d\theta}=r\,a^n\cos n\theta \Rightarrow a^n\sin n\theta\frac{dr}{d\theta}=r\,a^n\cos n\theta,$$

$$\therefore \frac{dr}{d\theta}=r\cot n\theta$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ to get orthogonal trajectory, we get,

$$-r\frac{d\theta}{dr}=\cot n\theta, \text{ on integration, we have, } r^n=b^n\cos n\theta,$$

which is the required equation of the orthogonal trajectories.

Example 21: Find the orthogonal trajectories of the family of parabolas $y=a\,x^2$.

Solution: We have

$$y=a\,x^2, \textit{differentiating w.r.to } x, \textit{ we get, } p=2\,a\,x$$

$$\text{Eliminating } a \textit{ from these two equations, we get, } y=\frac{p}{2\,x}x^2$$

$$\Rightarrow p\,x-2\,y=0$$

The orthogonal trajectory of the family of parabolas has the differential equation

$$\left(\frac{-1}{p}\right)x-2\,y=0 \Rightarrow x+2\,y\,p=0 \Rightarrow x\,dx+2\,y\,dy=0$$

$$\text{which on integration gives, } \frac{x^2}{2}+y^2=C$$

The above equation represents a family of ellipses and is the required equation of the orthogonal trajectory.

Example 22: Find the orthogonal trajectories of the family of rectangular hyperbola $x\,y=c$.

Solution: $\textit{We have, } x\,y=c, \textit{ differentiating w.r.to } x, \textit{ we get, } x\,p+y=0$

The orthogonal trajectory of the family has the differential equation

$$\left(\frac{-1}{p}\right)x+y=0 \Rightarrow -x+y\,p=0 \Rightarrow -x\,dx+y\,dy=0$$

$$which\,on\,\text{integration gives,}\ \frac{-x^2}{2}+\frac{y^2}{2}=C \Rightarrow (y-x)(y+x)=c,\ where\,c=2C$$

I.Q. 18

I.Q. 19

I.Q. 20

I.Q. 21

8. Existence and Uniqueness of Solutions of first order differential equations:

We have used different methods to get the solutions of an ordinary differential equation. While solving the differential equations we have observed that a solution may exist or may not exist. We also observed that if a solution exists it may be unique or may not be unique.

We will examine under what condition does the solution of a differential equation exists and is unique? Let us consider the first order differential equation

$$\frac{dy}{dt}=f(\mathrm{y},t),\ y(t_0)=y_0 \qquad (18)$$

Now the question arise that what are the conditions for an initial value problem (18) to have at least one solution? Also, we would like to know that what are the conditions for the problem (18) to have a unique solution.

The answers of the above questions are given by the **Existence Uniqueness Theorem.**

Theorem: **(Existence-Uniqueness Theorem)**:

If the function f(y, t) is continuous at every points (y, t) in some rectangle $D: |y-y_0|<a,\ |t-t_0|<b$ and bounded in D,

$$\left|f(y,t)\right| \le k \ \forall (y,t) \in D \qquad (19),$$

Then eqn. (18) has at least one solution y(t) in the interval $|t-t_0|<h$, where h is the smaller of the two numbers a and b/k. Also, if $\frac{\partial f}{\partial t}$ is continuous for all (y, t) in the rectangle D and bounded therein, $\left|\frac{\partial f}{\partial t}\right| \le M \ \forall (y,t) \in D$ then the solution y(t) is unique in the interval $|t-t_0|<h$.

Value Addition:
Remark: Since y' = f(y, t), then from eqn. (19), we have $|y'| \le m$. In other words the slope of the solution curve y(t) in D is at least - m and at most m. Hence we can say that a solution curve that crosses through any point (x_0, y_0) must lie in the region bounded by the lines having slopes - m and m, respectively.

Example 23: Examine whether the solution of the IVP, $\frac{dy}{dx} = \sqrt{|y|} \ when \ y(0)=0$, exist and unique or not.

Solution: We have $f(x,y) = \frac{dy}{dx} = \sqrt{|y|}, x_0 = 0 \ and \ y_0 = 0$. Let us take the rectangle D with $|x-0|<a \ and \ |y-0|<b$ where a and b any positive numbers. Since $f(x, y)$ is continuous and bounded in the rectangle D which contains $(0, 0)$. Therefore the solution of the IVP exists. For uniqueness consider the condition

$$\frac{\left|f(x,y_2)-f(x,y_1)\right|}{|y_2-y_1|} = \frac{\left|\sqrt{|y_2|}-\sqrt{|y_1|}\right|}{|y_2-y_1|}.$$

Lipschitz condition is violated for the region that contains y = 0. Since for $y_1 = 0 \ and \ y_2 > 0$ we have

$$\frac{|f(x,y_2)-f(x,y_1)|}{|y_2-y_1|}=\frac{\sqrt{y_2}}{y_2}=\frac{1}{\sqrt{y_2}},\, y_2>0.$$

and we can made it large enough as of our choice by choosing y_2 sufficiently small, whereas condition

$$|f(x,y_2)-f(x,y_1)|\le|y_2-y_1|$$

needs that the quotient on the left-hand side of the above equation does not exceed M (a fixed constant).

Therefore, we can say that the solution is not unique.

Example 24: Examine $\frac{dy}{dx}=y \; when \; y(0)=0$, for existence and uniqueness of solutions.

Solution: We have $f(x,y)=\frac{dy}{dx}=y,\, f_y(x,y)=1,\, x_0=0 \, and \; y_0=0$

Let us take the rectangle D defined by

$$D: |x-0|<a \, and \, |y-1|<b$$

where the numbers a and b are positive.

In the rectangle D that contains the point (0, 1), f(x, y) is continuous and bounded therein. Therefore the solution of the IVP exists. Also we have $f_y(x,y)=1$ is continuous and bounded in D. Therefore, we can say that the solution is unique.

Example 25: Examine $\frac{dy}{dx}=f(x,y)=\begin{cases} y(1-2x), \; for \; x>0 \\ y(2x-1), \; for \; x<0 \end{cases} \; with \; y(1)=1$, for existence and uniqueness of solutions.

Solution: The function $\frac{dy}{dx}=f(x,y)=\begin{cases} y(1-2x), \; for \; x>0 \\ y(2x-1), \; for \; x<0 \end{cases}$ is continuous and bounded everywhere except x = 0. Hence the solution exist and unique.

I.Q. 22

I.Q. 23

I.Q. 24

Summary:

1. The most general form of the first order and nth degree differential equation of is

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + \ldots + A_n = 0$$

where A_1, A_2,..., A_n are functions of the variables x and y.

2. If we resolve the eqn. $p^n + A_1 p^{n-1} + A_2 p^{n-2} + \ldots + A_n = 0$ into rational linear factors of the first order, then the above equation may be written as

$$(p - f_1)(p - f_2)\ldots(p - f_n) = 0$$

where f_1, f_2,..., f_n are functions of x and y.

If the solutions desired for eqn. $(p - f_1)(p - f_2)\ldots(p - f_n) = 0$ are

$$F_r(x, y, C_r) = 0, r = 1, 2, \ldots, n$$
$$\text{where } C_r,\ r = 1, 2, \ldots, n \ \textit{are arbitrary} \text{ constants}.$$

Then the general solution of eqn. $p^n + A_1 p^{n-1} + A_2 p^{n-2} + \ldots + A_n = 0$ is

$$F_1(x, y, C_1).F_2(x, y, C_2)\ldots F_n(x, y, C_n) = 0,$$

3. If the factorization of the eqn. $p^n + A_1 p^{n-1} + A_2 p^{n-2} + \ldots + A_n = 0$ into rational linear factors is not possible, then

a) we can solve it for x if it is expressed as $x = f(y, p)$

We have to differentiate the above equation with respect to y and try to solve the equation in the variables y and p. We get the required solution by eliminating p between the solution of resulting differential equation and given equation $x = f(y, p)$.

b) we can solve it for y if it is expressed as $y = g(\mathrm{x}, p)$

We have to differentiate the above equation with respect to x and try to solve the resulting differential equation in the variables x and p. We get the required solution by eliminating p between the solution of resulting differential equation and given equation $y = g(\mathrm{x}, p)$.

4. Clairaut's equation is given by $y = x\,p + f\left(p\right)$

We can solve this equation only for y and its solution is given by $y = xC + f\left(C\right)$

5. To get the singular solution of a differential equation, we need to find the envelope of the curves representing the general solution of the differential equation.

6. $F\left(x, y, -\frac{1}{\frac{dy}{dx}}\right) = 0$ is the equation of the orthogonal trajectory of the family of one parameter family of curves $f\left(x, y, C\right) = 0$. For a family of curves by the polar equation $f\left(\mathrm{r}, \theta, c\right) = 0$ the differential equation of the orthogonal trajectory is $F\left(\mathrm{r}, \theta, -\frac{r^2}{\frac{dr}{d\theta}}\right) = 0$.

7. The existence uniqueness theorem says that the sufficient condition of solution of the first order IVP $\frac{dy}{dt} = f\left(\mathrm{y}, t\right), y\left(t_0\right) = y_0$ in a region D defined by $D: \left|y - y_0\right| < a, \left|t - t_0\right| < b$ are

i) the function f is continuous and bounded in the region D.

Also if the solution of the IVP exists, then it is unique if, we have

ii) $\frac{\partial f}{\partial y}$ is continuous and bounded in D.

Exercise:

1. Solve the following first order differential equations for p.

$a)\ p^2 y + p x - p y - x = 0$

b) $p^2 - 7p + 12 = 0$

$c)\ p^3 = a x^4$

2. Solve the following equations first order differential equations for x.

$a)\ x = y + b \ln|p|$ b) $p^2 - p y + x = 0$ $c)\ x = y + p^2$

$d)\ y = x + b \tan^{-1} p$ $e)\ p^3 + p = e^y$

3. Solve the following equations first order differential equations for y.

$a)\ y = x p + p^2$

b) $y = x p + p - p^2$

$c)\ y = x^4 p^2 - p x$

.

4. Find the singular solution of the differential equation

$a)\ y = x p + \dfrac{1}{p}$

$b)\ y = x p + p^2$

5. Show that the system of confocal conics $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1$ is self-orthogonal.

6. Find the orthogonal trajectories of $\dfrac{2a}{r} = 1 + \cos\theta$ for varying a.

7. Examine $\dfrac{dy}{dx} = f(x,y) = \begin{cases} \dfrac{4x^3 y}{x^4 + y^2}, & for\ x \neq 0\ and\ y \neq 0 \\ 0, & for\ x = 0\ and\ y = 0 \end{cases}$ $with\ y(0) = 0$, for existence and uniqueness of solutions.

8. Examine the ordinary differential equation

$\dfrac{dy}{dx} = \begin{cases} 0 & for\ x < 0 \\ 1 & for\ x \geq 0 \end{cases}$, $\forall x \in R$ for existence of solution.

9. Does the equation y(x) $\dfrac{dy}{dx} = -e^{-y} x$ have a unique solution?

Existence and Uniqueness of Solutions of first order differential equations

Reference:

1. Shipley L. Ross – differential equation, third edition, John Wiley and sons , 1984
2. Edwards and Penney – Differential Equations and boundary value problem, computing and modeling, 3rd edition, Pearson Prentice Hall, 2004
3. Martha L. Abell and James P. Braselton, Differential Equations with Mathematica, A P professional, 1993
4. George F. Simmons - Differential Equations with application and historical notes, Tata McGraw Hills Publishing Company Ltd. 1979.